Energy Considerations for Problems Where Angular Momentum is conserved

Ladislav Kocis
Ladislav.Kocis@qr.com.au
Julius Kruttschnitt Mineral Research Centre, the University of Queensland, Isles Road, Indooroopilly Queensland 4068 Australia

(Received: 15.08.2011 Accepted: 13.10.2011)

Abstract
When the external torque acting on a rotating object is zero, its angular momentum is conserved. For a rotating object the angular momentum can be expressed as the product of rotational inertia and angular velocity. This gives an equation that is useful in problem solving. However, when using this equation there is no clue as to what is happening to the energy of the system. When introduced to the topic of conservation of angular momentum, students ask questions about the conservation of energy and kinetic energy. However, there is neither a textbook nor article in physics teaching journals that would address the issue of energy for cases where the angular momentum is conserved. The answers to questions of students are given here.
Key Words: Rotational dynamics, angular momentum, conservation of angular momentum, conservation of energy
PACS Numbers: 46.02C, 46.03B, 46.05B

Conservation of angular momentum

The equation of motion for a rotating object, or a body can be written as,

$$\mathbf{T}_{ext} = \frac{d\mathbf{L}}{dt}, \qquad (1)$$

where $\mathbf{T}_{ext}$ is the external torque, $\mathbf{L}$ is angular momentum and t stands for time. Equation (1) shows that if the external torque $\mathbf{T}_{ext}$ is zero then angular momentum $\mathbf{L}$ is constant. If a point object rotates e.g. about the z-axis, equation (1) gives

$$const = L_z = I\omega = mr^2\omega, \qquad (2)$$

where L_z is the component of the angular momentum along the rotation axis, I is the rotational inertia for the same axis, r is the object distance from the same axis and m is the object's mass. Eq. (2) implies that if eg. I increases then ω decreases, and conversely. Denoting the initial values of rotational inertia and angular velocity as I_0 and ω_0, respectively, the law of conservation of angular momentum can be written as

$$I_0\omega_0 = I\omega . \tag{3}$$

Let us turn to a simple example given by Haliday and Resnick (1981). In this example a small object of mass m is attached to a light string, which passes through a hollow tube. One hand and the string hold the tube by the other. The object is set into rotation in a circle of radius r_1 with speed v_1. The string is then pulled down which shortens the radius of the object's path to r_2. The task is to find the new speed v_2 and the new angular speed ω_2 of the object in terms of the initial values v_1 and ω_1 and the two radii.

Equation (3) requires that the initial angular momentum equals the final angular momentum, whence

$$mr_1^2\omega_1 = mr_2^2\omega_2 . \tag{4}$$

Given r_1, ω_1 and r_2 we can determine ω_2 from equation (4), find v_2 from $v_2 = \omega_2 r_2$, and the problem is solved. While this is quite clear someone may interested to know that what is happening with energy. The first question here is this: Is kinetic energy conserved? In the example in Fig. 1 and all the other similar problems, the rotational kinetic energy is not conserved. If $r_1 > r_2$, then $r_1^2 > r_2^2$ and equation (4) imply the inequality $\omega_1 < \omega_2$. This inequality multiplied by equation (4) gives

$$\frac{1}{2}mr_1^2\omega_1^2 = \frac{1}{2}I_1\omega_1^2 < \frac{1}{2}I_2\omega_2^2 = \frac{1}{2}mr_2^2\omega_2^2 . \tag{5}$$

Inequality (5) confirms that the (rotational) kinetic energy is not conserved. The second question to be answered is whether the overall energy is conserved. The answer is simple - it has to be. It is obvious that the amount of work supplied to the system is strongly dependent on the friction. Is this the reason that the energy changes are intractable? Or, is there other reason that would prevent us to see the energy changes in the system in Fig.1? These questions are answered in the following section.

Evaluation of the work done on the system

When the string in Fig. 1 is pulled down by dr the kinetic energy of the rotating object increases by,

$$d\left(\frac{1}{2}mr^2\omega^2\right) = d\left(\frac{1}{2}I\omega^2\right). \qquad (6)$$

At the same time pulling the string down means that a work was done on the system. By definition this work is

$$dW = -Fdr . \qquad (7)$$

The negative sign in the equation above appears because the radius decreases and thus its differential is negative. The force in the equation above is the sum of the two forces, namely F_1 and F_2. Force F_2 represents the force of friction and F_1 is the tension in the string, i.e. the centripetal force. If there is no friction then the work $-F_1dr$ has to be equal to the increase in the kinetic energy, which can be written as

$$d\left(\frac{1}{2}mr^2\omega^2\right) = -F_1dr . \qquad (8)$$

When the friction between the pipe and the string is not zero then F_2 is not zero and the expression $-F_2dr$ represents work done by friction so it is possible to write

$$dW_f = -F_2dr , \qquad (9)$$

where the symbol W_f stands for the work done by F_2. F_2. In example in Fig. 1 the work done by friction forces implies an increase in the thermal energy of the system.

The equation above is important but it does not need to be considered further on as we are not interested in the amount of the thermal energy that is being generated. It is not possible that F_2 also contributes to the increase of the kinetic energy of the rotating body. When friction acts, the product on the right-hand side of (9) is always converted to thermal energy.

As already mentioned the force F_1 in (8) has the magnitude of the centripetal force. Therefore we write

$$F_1 = m\omega^2 r . \qquad (10)$$

Now (10) is used in (8). The left side of (8) is then simplified using identity

$$(1/2)d\left(r^2\omega^2\right) = \omega^2 rdr + r^2\omega d\omega , \qquad (11)$$

and equation (8) becomes

$$\omega dr + r d\omega = -\omega dr. \tag{12}$$

Equation (12) can be rearranged as

$$\frac{d\omega}{\omega} = -2\frac{dr}{r}. \tag{13}$$

Let us recall now that initially $r = r_1$, $\omega = \omega_1$, and that upon the completion of the string movement $r = r_2$ and $\omega = \omega_2$. Integrating equation (13) from the initial state to the final state using formula

$$\int \frac{dr}{r} = \ln r, \quad r > 0 \tag{14}$$

and then applying rules for manipulating logarithmic functions we get

$$\frac{\omega_1}{\omega_2} = \frac{r_2^2}{r_1^2}. \tag{15}$$

Equation (15) is equivalent to equation (4). This means that equation (4) can be derived not just by applying $\mathbf{T}_{ext} = \mathbf{0}$ in (1), but also by considering the work done on the system and the subsequent energy increase. The procedure starting with (6) shows that while the kinetic energy of the object is not conserved we can easily describe energy changes. Such approach shows the problem in a dynamic process where radius, angular velocity and linear velocity are all changing. When the force pulls the string in Fig. 1 the mass moves towards the hollow tube. The angular velocity, radius and the kinetic energy are all being changed. To see that the rotational kinetic energy is increasing is obvious from equation (8). The change in ω as a response to the change in r is seen in equation (13). The latter equation can be read like this: The relative increase in ω equals the double of the corresponding relative decrease in r. Finally, the integration of (13) gives (15), and (15) assures that while everything varied in the process, the angular momentum remained constant as required by (1), because the external torque was zero.

Final Notes and Conclusions

Writing this article was inspired by questions of students. Occasionally some of them think that in examples like the one in Fig.1 we should use the conservation of energy. It is the lecturer, who upon hearing such statement tends to rectify it by saying: "You cannot do it in that way because the kinetic energy is not conserved." This leaves students wandering that what is actually happening to the conservation of energy because the overall energy has to be conserved. The important moment here is that the work done on the rotating object can be evaluated and changes in the energy of the system can be understood.

The considerations presented in previous sections give a deeper insight into the mechanics of rotating bodies at conditions when the external torque is zero. They show how rotational energy, work and angular momentum are related. Finally, the ideas and derivation given here help to appreciate the simplicity and elegance that is contained in equations expressing the conservation of the angular momentum. The reader who is interested to review energy changes in a different example may read the Appendix A.

Conservation of angular momentum as a topic is well mastered in university textbooks of physics and tutorials see e.g. (Halliday and Resnick, 1981; Knight, 2004; McDermott et al., 2002). Publications reporting on teaching angular momentum and its conservation show new approaches and methodologies, see e.g. (McDermott et al., 2002; Close and Heron, 2011; Lock, 1989; Johns, 1998) and new original experiments, see e.g. (Johns, 1998; Johns, 2003; Williamson et al., 2000; Mak and Wong, 1989; Klostergaard, 1976; Rockefeller, 1975). However, while quality contributions that relate to teaching of angular momentum and its conservation can be found in physics teaching journals, one aspect of this topic appears to be neglected. It is the energy consideration for cases where the angular momentum is conserved. So besides answering questions of students the purpose of this article is to initiate such discussion.

References

Close, H., G., Heron, P., R. (2011). L. Student understanding of the angular momentum of classical particles. *Am. J. Phys.* 79(10), 1068-1078.

Halliday, D. and Resnick, R. (1981). *Fundamentals of Physics*, 2nd ed. John Wiley & Sons, Inc.: New York, pp. 193-196.

Johns, R. H., (1998). Physics on a Rotating Reference Frame, *The Phys. Teach.* 36(3), 178-179.

Palmieri, J., N. and Strauch, K. (1963). An experiment on angular momentum for the introductory laboratory. *Am. J. Phys*. 31, 91-95.

Klostergaard, H. (1976). Conservation of angular momentum – a demonstration. *Am. J. Phys.* 44(1), 21-24.

Knight, R., D. (2004). *Physics for scientists and Engineers: A Strategic Approach with modern Physics.* Pearson/Addison-Wesley: San Francisco, 396-401.

Lock, J., A. (1989). An alternative approach to the teaching of rotational dynamics. *Am. J. Phys.* 57(5), 428-434.

Mak, S., Y. and Wong, K., Y. (1989). A qualitative demonstration of the conservation of angular momentum in a system of two non-coaxial rotating discs. *Am. J. Phys.* 57(10), 951-956.

McDermott, L., C., Shaffer, P., S. and the Physics Education Group at the University of Washington (2002). *Tutorials in Introductory Physics*. Prentice Hall: Upper Saddle River, NJ.

Rockefeller R., R. (1975). Conservation of angular and linear momentum on an air table. *Am. J. Phys.* 43(11), 981-987.

Williamson, J., C., Torres-Isea R., O. and Kletzing, C., A. (2000). Analyzing linear and angular momentum conservation in digital videos of puck collisions. *Am. J. Phys.* 68(9), 841-847.

Appendix

In this paper, we consider a case where the rotational inertia has two parts. One part of the inertia is constant and the other varies. This problem can be illustrated with another example taken from reference [1]. A student sits on a stool that is free to rotate about a vertical axis. The student holds arms extended horizontally with a dumbbell in each hand. Mass of the dumbbells is m. The instructor starts rotating student with an angular speed ω_1. Assume that friction is negligible and exerts no torque about the vertical axis of rotation. Assume also that the rotational inertia of the student I remains constant as he pulls his hands to his sides and that the change in total rotational inertia is due only to pulling the dumbbells in. The original distance of the dumbbells from the axis of rotation is denoted as r_1 and the final as r_2. The task is to find the final angular speed of the student, or better to say of the whole system.

The conservation of energy offers equation

$$d\left(\frac{1}{2}I\omega^2\right)+d\left(\frac{1}{2}mr^2\omega^2\right)=-m\omega^2 rdr. \tag{16}$$

As in the previous example the $m\omega^2 r$ on the right-hand side of equation (16) is the centripetal force, in this case exerted by the student on the dumbbells. The left-hand side of equation (16) includes the change in the kinetic energy of both the student and the dumbbells.
The differentials in the left side of this equation can be expressed in terms of $d\omega$ and dr while I remains constant. Equation (16) then simplifies and has form

$$I\omega d\omega + mr^2\omega d\omega = -2m\omega^2 rdr. \tag{17}$$

This equation allows separation of the two variables, namely ω and r.

$$\frac{d\omega}{\omega} = -2\frac{rdr}{I/m + r^2}. \tag{18}$$

Now the task is to integrate (18). For this purpose it is advantageous to use substitution

$$r^2 = z, \qquad 2rdr = dz \tag{19}$$

and write down

$$\int_{\omega_1}^{\omega_2} \frac{d\omega}{\omega} = -\int_{z_1}^{z_2} \frac{dz}{I/m + z}. \tag{20}$$

where

$$z_1 = r_1^2, \qquad z_2 = r_2^2. \tag{21, 22}$$

The first integral in (20) is evaluated using formula (14). The second integral is evaluated using formula

$$\int \frac{dz}{I/m + z} = \ln(I/m + z), \quad r > 0. \tag{23}$$

In this way we get relation

$$\frac{\omega_2}{\omega_1} = \frac{I/m + r_1^2}{I/m + r_2^2}, \tag{24}$$

this can be rewritten as

$$\left(I + mr_1^2\right)\omega_1 = \left(I + mr_2^2\right)\omega_2. \tag{25}$$

Equation (25) expresses the conservation of angular momentum. As stated in the example at the beginning of this section quantities I, m, r_1, r_2, and ω_1 are known, hence (25) can be used for calculation of ω_2.

Figures

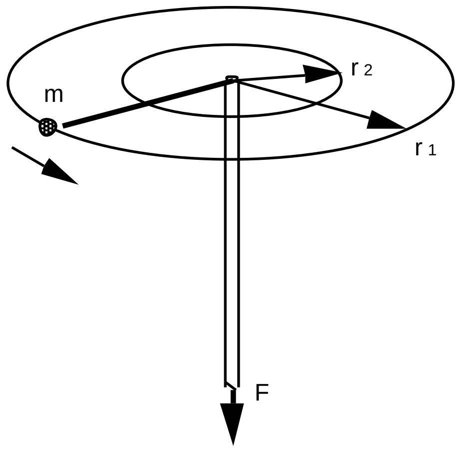

Fig. 1. An object attached to a string is set into motion in a horizontal plane. The string that is passing down through a hollow tube is pulled down to decrease the radius of rotation