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The Kadomtsev-Petviashvili (KP) Equation as an example of Invariant Equations

Invariant denklemlerine bir örnek: Kadomtsev-Petviashvili (KP) denklemi

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Abstract

We will consider two scalar differential operators L and M. We call them scalar because their coefficients are functions rather than matrices. Our aim is to obtain some gauge invariant equations. In order to achieve this goal we firstly find the gauge invariants of L and M by using the gauge transformation on L and M. And then by using the Lax equation, $[L, M] = 0$, we have some differential equations in the invariant forms. After that we derive The Kadomtsev-Petviashvili (KP) equation as an example.

Key words: KP equation

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Ozet

L ve M'yi iki skaler diferansiyel operatör olarak gözönüne alalım. Burada L ve M skaler operatörlerdir, çünkü bu operatörlerin katsayıları matrisler değil fonksiyonlardır. Bizim amacımız 'gauge invariant' denklemlerini elde etmektir. Bu amaca ulaşmak için ilk önce L ve M üzerinde 'gauge transformasyon' kullanarak L ve M nin 'gauge invariant' larını bulacağız. Daha sonra Lax denklemini, $[L, M] = 0$, kullanarak invariant form'da bazı diferansiyel denklemler elde edeceğiz. Burada Kadomtsev-Petviashvili (KP) denklemi bir örnek olarak karşımıza çıkacaktır.

Anahtar Kelimeler: KP denklemi

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1 Introduction

Let L and M be two operator functions such that

$$L = \partial_t + \partial_x^2 + u\partial_x + v \quad (1)$$

$$M = \partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c \quad (2)$$

where u, v, a, b and c are functions of x, y and t.

Consider the commutativity representation $[L, M] = 0$, i.e. L and M are a Lax pair [2].

2 Invariants for L and M

We will use the gauge transformation [1] on L and M in order to get invariants for L and M.

2.1 Invariants for $L = \partial_t + \partial_x^2 + u\partial_x + v$

$$g^{-1}Lg = \tilde{L}$$

Therefore,

$$g^{-1}(\partial_t + \partial_x^2 + u\partial_x + v)g = \partial_t + \partial_x^2 + \tilde{u}\partial_x + \tilde{v} \Rightarrow$$

$$g^{-1}(\partial_t.g) + g^{-1}(\partial_x^2.g) + ug^{-1}(\partial_x.g) + v = \partial_t + \partial_x^2 + \tilde{u}\partial_x + \tilde{v} \Rightarrow$$

$$g^{-1}(g_t+g\partial_t)+g^{-1}(g_{xx}+2g_x\partial_x+g\partial_x^2)+ug^{-1}(g_x+g\partial_x)+v = \partial_t+\partial_x^2+\tilde{u}\partial_x+\tilde{v} \Rightarrow$$

$$\partial_t+\partial_x^2+(u+2g^{-1}g_x)\partial_x+v+g^{-1}g_t+ug^{-1}g_x+g^{-1}g_{xx} = \partial_t+\partial_x^2+\tilde{u}\partial_x+\tilde{v}$$

Therefore, we obtain

$$\tilde{u} = u + 2g^{-1}g_x \quad (3)$$

$$\tilde{v} = v + g^{-1}g_t + ug^{-1}g_x + g^{-1}g_{xx} \quad (4)$$

So,

$$\tilde{u} = u + 2g^{-1}g_x \Rightarrow$$

$$g^{-1}g_x = \frac{1}{2}(\tilde{u} - u) \tag{5}$$

In the equation (5), by taking derivative both sides with respect to x, we obtain

$$g^{-1}g_{xx} = \frac{1}{2}(\tilde{u} - u)_x + (g^{-1}g_x)^2$$

If we substitute the equation (5) in the above equation, we have

$$g^{-1}g_{xx} = \frac{1}{2}(\tilde{u} - u)_x + \frac{1}{4}(\tilde{u} - u)^2$$

Therefore, by substituting $g^{-1}g_x$ and $g^{-1}g_{xx}$ in the equation (4), we have

$$g^{-1}g_t = \tilde{v} - v - \frac{1}{2}(\tilde{u} - u)_x + \frac{1}{4}(u^2 - \tilde{u}^2)$$

Since $(g^{-1}g_x)_t = (g^{-1}g_t)_x$,

$$\frac{1}{2}(\tilde{u} - u)_t = \tilde{v}_x - v_x - \frac{1}{2}(\tilde{u} - u)_{xx} + \frac{1}{2}(uu_x - \tilde{u}\tilde{u}_x)$$

By multiplying both sides by 2, we obtain

$$\tilde{u}_t + \tilde{u}_{xx} + \tilde{u}\tilde{u}_x - 2\tilde{v}_x = u_t + u_{xx} + uu_x - 2v_x$$

Let

$$J = u_t + u_{xx} + uu_x - 2v_x \tag{6}$$

This is an invariant for the operator function L [3], where L is defined in (1).

2.2 Invariants for $M = \partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c$

$$g^{-1}Mg = \widetilde{M}$$

Therefore,

$$g^{-1}(\partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c)g = \partial_y + \partial_x^3 + \tilde{a}\partial_x^2 + \tilde{b}\partial_x + \tilde{c} \Rightarrow$$

$$g^{-1}(\partial_y.g)+g^{-1}(\partial_x^3.g)+ag^{-1}(\partial_x^2.g)+bg^{-1}(\partial_x.g)+c = \partial_y+\partial_x^3+\tilde{a}\partial_x^2+\tilde{b}\partial_x+\tilde{c} \Rightarrow$$

$$g^{-1}(g_y+g\partial_y)+g^{-1}(g_{xxx}+3g_{xx}\partial_x+3g_x\partial_x^2+g\partial_x^3)+ag^{-1}(g_{xx}+2g_x\partial_x+g\partial_x^2)$$

$$+bg^{-1}(g_x+g\partial_x)+c = \partial_y+\partial_x^3+\tilde{a}\partial_x^2+\tilde{b}\partial_x+\tilde{c} \Rightarrow$$

$$\partial_y+\partial_x^3+(a+3g^{-1}g_x)\partial_x^2+(b+2ag^{-1}g_x+3g^{-1}g_{xx})\partial_x+c+g^{-1}g_y+bg^{-1}g_x$$

$$+ag^{-1}g_{xx}+g^{-1}g_{xxx} = \partial_y+\partial_x^3+\tilde{a}\partial_x^2+\tilde{b}\partial_x+\tilde{c}$$

So, we obtain

$$\tilde{a} = a+3g^{-1}g_x \tag{7}$$

$$\tilde{b} = b+2ag^{-1}g_x+3g^{-1}g_{xx} \tag{8}$$

$$\tilde{c} = c+g^{-1}g_y+bg^{-1}g_x+ag^{-1}g_{xx}+g^{-1}g_{xxx} \tag{9}$$

Now, from the equation (7), we have

$$g^{-1}g_x = \frac{1}{3}(\tilde{a}-a) \tag{10}$$

In the equation (10), by taking derivative both sides with respect to x, we obtain

$$g^{-1}g_{xx} = \frac{1}{3}(\tilde{a}-a)_x+(g^{-1}g_x)^2$$

If we substitute $g^{-1}g_x$ (10) in the above equation, we get

$$g^{-1}g_{xx} = \frac{1}{3}(\tilde{a}-a)_x+\frac{1}{9}(\tilde{a}-a)^2 \tag{11}$$

We know that $\tilde{b} = b+2ag^{-1}g_x+3g^{-1}g_{xx}$ (8)

By substituting $g^{-1}g_x$ (10) and $g^{-1}g_{xx}$ (11) in the equation (8), we have

$$\tilde{b} = b+\frac{2}{3}a(\tilde{a}-a)+(\tilde{a}-a)_x+\frac{1}{3}(\tilde{a}-a)^2 \Rightarrow$$

$$\tilde{a}_x+\frac{1}{3}\tilde{a}^2-\tilde{b} = a_x+\frac{1}{3}a^2-b \tag{12}$$

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Let

$$I = a_x + \frac{1}{3}a^2 - b \tag{13}$$

I is an invariant for the operator function M, where M is defined in (2). Now, consider the equation (9), which is

$$\tilde{c} = c + g^{-1}g_y + bg^{-1}g_x + ag^{-1}g_{xx} + g^{-1}g_{xxx}$$

In the equation (11), if we take derivative both sides w.r.t. x, we have

$$g^{-1}g_{xxx} = \frac{1}{3}(\tilde{a} - a)_{xx} + \frac{2}{9}(\tilde{a} - a)(\tilde{a} - a)_x + (g^{-1}g_x)(g^{-1}g_{xx})$$

By substituting $g^{-1}g_x$ (10) and $g^{-1}g_{xx}$ (11) in the above equation, we obtain

$$g^{-1}g_{xxx} = \frac{1}{3}(\tilde{a} - a)_{xx} + \frac{1}{3}(\tilde{a} - a)(\tilde{a} - a)_x + \frac{1}{27}(\tilde{a} - a)^3 \tag{14}$$

If we substitute $g^{-1}g_x$ (10), $g^{-1}g_{xx}$ (11) and $g^{-1}g_{xxx}$ (14) in the equation (9),
we get the following expression

$$\tilde{c} = c + g^{-1}g_y + \frac{1}{3}\tilde{a}b - \frac{1}{3}ab + \frac{1}{3}\tilde{a}(\tilde{a} - a)_x + \frac{1}{9}a(\tilde{a} - a)^2 + \frac{1}{3}(\tilde{a} - a)_{xx} + \frac{1}{27}(\tilde{a} - a)^3$$

From the equation (12), we can easily obtain

$$b = a_x + \frac{1}{3}a^2 - \tilde{a}_x - \frac{1}{3}\tilde{a}^2 + \tilde{b}$$

By substituting b in the $\frac{1}{3}\tilde{a}b$ term, the equation above becomes

$$\tilde{c} = c + g^{-1}g_y + \frac{1}{3}(\tilde{a} - a)_{xx} - \frac{1}{3}ab + \frac{1}{3}\tilde{a}\tilde{b} + \frac{2}{27}a^3 - \frac{2}{27}\tilde{a}^3$$

By multiplying 3 and then taking the derivative with respect to x,
the equation above becomes

$$3\tilde{c}_x = 3c_x + 3(g^{-1}g_y)_x + (\tilde{a}_{xx} - \frac{2}{9}\tilde{a}^3 + \tilde{a}\tilde{b})_x - (a_{xx} - \frac{2}{9}a^3 + ab)_x \tag{15}$$

Applying equation (10), which is

$$g^{-1}g_x = \frac{1}{3}(\tilde{a} - a)$$

By taking the derivative both sides w.r.t y, we have

$$(g^{-1}g_x)_y = \frac{1}{3}(\tilde{a} - a)_y$$

Since $(g^{-1}g_x)_y = (g^{-1}g_y)_x$,

$$(g^{-1}g_y)_x = \frac{1}{3}(\tilde{a} - a)_y$$

So,

if we substitute this last expression above in the equation (15), we obtain

$$\tilde{a}_y + (\tilde{a}_{xx} - \frac{2}{9}\tilde{a}^3 + \tilde{a}\tilde{b})_x - 3\tilde{c}_x = a_y + (a_{xx} - \frac{2}{9}a^3 + ab)_x - 3c_x$$

Let

$$K = a_y + (a_{xx} - \frac{2}{9}a^3 + ab)_x - 3c_x \tag{16}$$

where K is an invariant for the operator function M.

Summarise all invariants for the operator functions L and M:

$$\begin{aligned} J &= u_t + u_{xx} + uu_x - 2v_x \\ I &= a_x + \frac{1}{3}a^2 - b \\ K &= a_y + (a_{xx} - \frac{2}{9}a^3 + ab)_x - 3c_x \end{aligned}$$

3 The Commutativity Equations

$$[L, M] = 0 \Rightarrow$$

$$[\partial_t + \partial_x^2 + u\partial_x + v, \partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c] = 0 \Rightarrow$$

$$(\partial_t + \partial_x^2 + u\partial_x + v)(\partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c) - (\partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c)(\partial_t + \partial_x^2 + u\partial_x + v) = 0$$

By doing some calculations, we have

$c_t + c_{xx} + uc_x - v_y - v_{xxx} - av_{xx} - bv_x + (b_t + b_{xx} + 2c_x + ub_x - u_y - u_{xxx} - 3v_{xx} - au_{xx} - 2av_x - bu_x)\partial_x + (a_t + a_{xx} + 2b_x + ua_x - 3u_{xx} - 3v_x - 2au_x)\partial_x^2 + (2a_x - 3u_x)\partial_x^3 = 0$

Therefore, we obtain the following equations:

$$2a_x - 3u_x = 0 \quad (17)$$

$$a_t + a_{xx} + 2b_x + ua_x - 3u_{xx} - 3v_x - 2au_x = 0 \quad (18)$$

$$b_t + b_{xx} + 2c_x + ub_x - u_y - u_{xxx} - 3v_{xx} - au_{xx} - 2av_x - bu_x = 0 \quad (19)$$

$$c_t + c_{xx} + uc_x - v_y - v_{xxx} - av_{xx} - bv_x = 0 \quad (20)$$

So,

$$(17) \Rightarrow 2a_x - 3u_x = 0 \Rightarrow$$

$a_x = \dfrac{3}{2}u_x \Rightarrow a = \dfrac{3}{2}u + \alpha(t, y)$, where α is an arbitrary function of t and y.

Let us choose $\alpha = 0$. Then, we have

$$a = \frac{3}{2}u \quad (21)$$

$$(18) \Rightarrow a_t + a_{xx} + 2b_x + ua_x - 3u_{xx} - 3v_x - 2au_x = 0$$

Since $I = a_x + \frac{1}{3}a^2 - b$ (13),

$$b = a_x + \frac{1}{3}a^2 - I \quad (22)$$

Therefore,

$$b_x = a_{xx} + \frac{2}{3}aa_x - I_x$$

If we substitute b_x in the equation (18), then the equation (18) becomes

$$a_t + 3a_{xx} + \frac{4}{3}aa_x - 2I_x + ua_x - 3u_{xx} - 3v_x - 2au_x = 0$$

In the above equation if we put $a = \frac{3}{2}u$ (21), then we obtain

$$\frac{3}{2}u_t + \frac{3}{2}u_{xx} + \frac{3}{2}uu_x - 2I_x - 3v_x = 0$$

Multiplying both sides by 2, we get

$$4I_x = 3(u_t + u_{xx} + uu_x - 2v_x)$$

Therefore, we obtain

$$I_x = \frac{3}{4}J \tag{23}$$

where $J = u_t + u_{xx} + uu_x - 2v_x$.

$$(19) \Rightarrow b_t + b_{xx} + 2c_x + ub_x - u_y - u_{xxx} - 3v_{xx} - au_{xx} - 2av_x - bu_x = 0$$

We already know that $b = a_x + \frac{1}{3}a^2 - I$ (22)
Since $a = \frac{3}{2}u$ (21),

$$b = \frac{3}{2}u_x + \frac{3}{4}u^2 - I \tag{24}$$

Therefore,

$$b_t = \frac{3}{2}(u_{xt} + uu_t) - I_t$$

and

$$b_x = \frac{3}{2}(u_{xx} + uu_x) - I_x \Rightarrow b_{xx} = \frac{3}{2}(u_{xxx} + u_x^2 + uu_{xx}) - I_{xx}$$

From equation (16), we have

$$c_x = \frac{1}{3}[a_y + (a_{xx} - \frac{2}{9}a^3 + ab)_x - K] \tag{25}$$

Since $a = \frac{3}{2}u$ and $b = \frac{3}{2}u_x + \frac{3}{2}u^2 - I$,

$$c_x = \frac{1}{2}u_y + \frac{1}{2}u_{xxx} + \frac{3}{8}u^2u_x + \frac{3}{4}u_x^2 - \frac{1}{2}u_xI + \frac{3}{4}uu_{xx} - \frac{1}{2}uI_x - \frac{1}{3}K \tag{26}$$

Hence, if we substitute $a(u)$, $b(u,I)$, $b_t(u,I)$, $b_x(u,I)$, $b_{xx}(u,I)$ and

$c_x(u, I, K)$ in the equation (19), we obtain

$$\frac{3}{2}u_{xt}+\frac{3}{2}uu_t-I_t+\frac{3}{2}u_{xxx}+\frac{3}{2}u_x^2+3uu_{xx}-I_{xx}+\frac{3}{2}u^2u_x-2uI_x-\frac{2}{3}K-3v_{xx}-3uv_x=0$$

$\Rightarrow$

$$\frac{3}{2}(u_t+u_{xx}+uu_x-2v_x)_x+\frac{3}{2}u(u_t+u_{xx}+uu_x-2v_x)-I_t-I_{xx}-2uI_x-\frac{2}{3}K=0$$

Since $J = u_t + u_{xx} + uu_x - 2v_x$ (6), the equation above becomes

$$\frac{3}{2}J_x + \frac{3}{2}uJ - I_t - I_{xx} - 2uI_x - \frac{2}{3}K = 0 \qquad (27)$$

So,
if we substitute $I_x = \frac{3}{4}J$ (23) in the equation (27), we obtain

$$K = \frac{9}{8}J_x - \frac{3}{2}I_t \qquad (28)$$

$$(20) \Rightarrow c_t + c_{xx} + uc_x - v_y - v_{xxx} - av_{xx} - bv_x = 0$$

In the equation (25), if we integrate both sides w.r.t. x, then we have

$$c = \frac{1}{3}(\int a_y dx + a_{xx} - \frac{2}{9}a^3 + ab - \int K dx) + \beta(t, y)$$

where β is an arbitrary function of t and y. Choose $\beta = 0$. Therefore,

$$c_t = \frac{1}{3}(\int a_{yt} dx + a_{xxt} - \frac{2}{3}a^2a_t + a_tb + ab_t - \int K_t dx)$$

Since $a = \frac{3}{2}u$ and $b = \frac{3}{2}u_x + \frac{3}{4}u^2 - I$ (24), the equation above becomes

$$c_t = \frac{1}{2}\int u_{yt}dx+\frac{1}{2}u_{xxt}+\frac{3}{8}u^2u_t+\frac{3}{4}u_tu_x-\frac{1}{2}u_tI+\frac{3}{4}uu_{xt}-\frac{1}{2}uI_t-\frac{1}{3}\int K_t dx$$

In the equation (26), if we take derivative of both sides w.r.t. x, then we can easily get

$$c_{xx} = \frac{1}{2}u_{xy}+\frac{1}{2}u_{xxxx}+\frac{3}{4}uu_x^2+\frac{3}{8}u^2u_{xx}+\frac{9}{4}u_xu_{xx}-\frac{1}{2}u_{xx}I-u_xI_x+\frac{3}{4}uu_{xxx}-\frac{1}{2}uI_{xx}-\frac{1}{3}K_x$$

Also from invariant $J = u_t + u_{xx} + uu_x - 2v_x$ (6), we can have

$$v_x = \frac{1}{2}(u_t + u_{xx} + uu_x - J) \tag{29}$$

By integrating the equation above w.r.t. x, we get

$$v = \frac{1}{2}(\int u_t dx + u_x + \frac{1}{2}u^2 - \int J dx) + \gamma(t, y)$$

where γ is an arbitrary function of t and y. Choose $\gamma = 0$, so that

$$v_y = \frac{1}{2}(\int u_{ty} dx + u_{xy} + uu_y - \int J_y dx)$$

In the equation (29), if we take derivative both sides w.r.t. x twice, then we obtain v_{xx} and v_{xxx} respectively:

$$v_{xx} = \frac{1}{2}(u_{xt} + u_{xxx} + u_x^2 + uu_{xx} - J_x),$$

$$v_{xxx} = \frac{1}{2}(u_{xxt} + u_{xxxx} + 3u_xu_{xx} + uu_{xxx} - J_{xx})$$

Hence, if we substitute $a, b, c_t, c_x, c_{xx}, v_y, v_x, v_{xx}$ and v_{xxx} in the equation (20), where $a = a(u), b = b(u, I), c = c(u, I, K)$ and $v = v(u, J)$ then we obtain

$-\frac{1}{3}K_x + \frac{1}{2}J_{xx} - \frac{1}{3}\int K_t dx + \frac{1}{2}\int J_y dx + \frac{3}{4}u_xJ + \frac{3}{8}u^2J - \frac{1}{2}IJ + \frac{3}{4}uJ_x - \frac{1}{2}u^2I_x - \frac{1}{3}uK - \frac{1}{2}uI_t - u_xI_x - \frac{1}{2}uI_{xx} = 0$

If we substitute $I_x = \frac{3}{4}J$ (23) in the above equation, then we obtain

$$-\frac{1}{3}K_x + \frac{1}{2}J_{xx} - \frac{1}{3}\int K_t dx + \frac{1}{2}\int J_y dx - \frac{1}{2}IJ + \frac{3}{8}uJ_x - \frac{1}{3}uK - \frac{1}{2}uI_t = 0$$

Similarly by substituting $K = \frac{9}{8}J_x - \frac{3}{2}I_t$ (28) in the last equation, then the equation above becomes

$$\frac{1}{2}\int J_y dx - \frac{1}{2}IJ + \frac{1}{2}J_{xx} - \frac{1}{3}K_x - \frac{1}{3}\int K_t dx = 0$$

In the above equation, if we take derivative of both sides with respect to x and multiply by '-1', then we obtain the following invariant equation

$$\frac{1}{3}K_t + \frac{1}{3}K_{xx} - \frac{1}{2}J_y - \frac{1}{2}J_{xxx} + \frac{1}{2}(IJ)_x = 0 \qquad (30)$$

Summarise all invariant equations:

$$\begin{aligned} I_x &= \frac{3}{4}J \\ K &= \frac{9}{8}J_x - \frac{3}{2}I_t \\ \tfrac{1}{3}K_t &+ \frac{1}{3}K_{xx} - \frac{1}{2}J_y - \frac{1}{2}J_{xxx} + \frac{1}{2}(IJ)_x = 0 \end{aligned}$$

4 Conclusion

If we put $I = -\frac{3}{2}q$ in the equation (23),$I_x = \frac{3}{4}J$, then we obtain

$$J = -2q_x \qquad (31)$$

Also, by substituting $I = -\frac{3}{2}q$ and $J = -2q_x$ in the equation (28), which is

$$K = \frac{9}{8}J_x - \frac{3}{2}I_t$$

Then we can rewrite K in the term of q

$$K = -\frac{9}{4}q_{xx} + \frac{9}{4}q_t \qquad (32)$$

Now, let us consider the invariant equation (30):

$$\frac{1}{3}K_t + \frac{1}{3}K_{xx} - \frac{1}{2}J_y - \frac{1}{2}J_{xxx} + \frac{1}{2}(IJ)_x = 0$$

Since $I = -\frac{3}{2}q$, $J = -2q_x$ (31) and $K = -\frac{9}{4}q_{xx} + \frac{9}{4}q_t$ (32), the equation above becomes

$$-q_{xy} - \frac{3}{2}q_x^2 - \frac{3}{2}qq_{xx} - \frac{1}{4}q_{xxxx} - \frac{3}{4}q_{tt} = 0$$

If we multiply both sides by '-4', then we easily get

$$4q_{xy} + 6q_x^2 + 6qq_{xx} + q_{xxxx} + 3q_{tt} = 0$$

$$\Rightarrow (4q_y + 6qq_x + q_{xxx})_x + 3q_{tt} = 0$$

By swapping y and t in the above equation, we finally obtain the following equation

$$(4q_t + 6qq_x + q_{xxx})_x + 3q_{yy} = 0 \qquad (33)$$

which is KP equation.

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